International Journal of Theoretical Physics, Vol. 1, No. 3 (1968), pp. 285–297.

Spectral Theory in Quantum Logics

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Abstract

If one supposes a quantum logic L to be a σ-orthocomplete, orthomodular partially ordered set admitting a set of σ-orthoadditive functions (called states) from L to the unit intervals [0, 1] such that these states distinguish the ordering and orthocomplement on L, then the observables on L are identified with L-valued measures defined on the Borel subsets of the real line. In this structure (and without the aid of Hilbert space formalism) the author shows that (1) the spectrum of an observable can be completely characterised by studying the observable $(A-\lambda)^{-1}$, and (2) corresponding to every observable A there is a spectral resolution uniquely determined by A and uniquely determining A.

1. *Introduction*

Until quite recently the observables in non-relativistic quantum mechanics have been identified with the set of self adjoint operators on a separate, infinite dimensional, complex Hilbert space. Likewise, on the same Hilbert space, the states have been identified with the trace operators of trace class 1. However, with the advent of Mackey's book on the mathematical foundations of quantum mechanics (Mackey, 1963), both observables and states have assumed a more abstract character having no overt connection with Hilbert space. This had led some investigators to consider the problem of deciding which quantum mechanical results are essentially consequences of Hilbert space formalism and which can be obtained without involving Hilbert space (Gudder, 1966; Ramsay, 1966; Varadarajan, 1962). In this paper we will show that most of the desirable theorems involving spectra can be obtained without the use of Hilbert space.

2. *Basic definitions*

By a *partially ordered set* (abbreviated *poset*) we will mean a pair $(P, \leqslant)$ such that P is a set and $\leqslant$ is a reflexive, antisymmetric, and transitive binary relation on P. $\leqslant$ is called a *partial order* on P. P is said to be *bounded* in case there exist (necessarily unique) elements 0 and 1 such that for all $x \in P$, $0 \leqslant x \leqslant 1$. If $M \subset P$, we say that M has an *infimum* (resp. *supremum*) providing there exists an element $m \in P$

(resp. $u \in P$) such that $x \in P \Rightarrow m \leqslant x$ (resp. $x \leqslant u$) and $s \leqslant x$ for every $x \in P$ (resp. $x \leqslant s$ for every $x \in P$) $\Rightarrow s \leqslant m$ (resp. $u \leqslant s$). If the infimum (resp. supremum) exists we then write $m = \wedge M$ (resp. $u = \vee M$). In case $M = \{a,b\}$ we write $\wedge$ and $\vee$ as infixes, i.e., $\wedge M = a \wedge b$ and $\vee M = a \vee b$.

By an *orthocomplemented poset* we will mean a triple $(P, \leqslant, ')$ such that $(P, \leqslant)$ is a bounded poset and $': P \to P$ is an order inverting involution such that for each $x \in P, x$ and x' are complements. $x, y \in P$ are said to be *orthogonal*, written $x \perp y$, providing $x \leqslant y'$. Note that $x \perp y$ if and only if $y \perp x$. An *orthomodular poset* is an orthocomplemented poset $(P, \leqslant, ')$ satisfying the additional conditions:

(1) if $\{x_1, x_2, \ldots, x_n\}$ is a mutually orthogonal family of elements of P, then

$$\bigvee_{i=1}^{n} \{x_i\}$$

exists, and

(2) if $x, y \in P$, $x \leqslant y$, then $y = x \vee (y \wedge x')$.

$x, y \in P$ are said to *commute*, written $x \mathrel{C} y$, providing there exist three mutually orthogonal elements $x_1, y_1, z \in P$ such that $x = x_1 \vee z$ and $y = y_1 \vee z$. It is a standard result that

$$x \mathrel{C} y \Leftrightarrow x' \mathrel{C} y \Leftrightarrow x \mathrel{C} y' \Leftrightarrow x' \mathrel{C} y'.$$

Also if x and y are comparable, then $x \mathrel{C} y$. The following theorem, due independently to D. J. Foulis and S. Holland, Jr., is most useful when working with orthomodular posets. Its proof is similar to that for lattices, and the latter can be found in Foulis (1962, p. 68, Theorem 5).

2.1. *Theorem*

Let P be an orthomodular poset, $a, b, c \in P$. Suppose that $a \vee b$, $a \wedge c$, $b \wedge c$, $(a \vee b) \wedge c$, and $(a \wedge c) \vee (b \wedge c)$ all exist in P, and that two of the three relations $a \mathrel{C} b$, $a \mathrel{C} c$, $b \mathrel{C} c$ hold. Then

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$

Also, assuming appropriate existence, this result together with the above remarks imply that

$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$

If P has the property that the join of a countable orthogonal family always exists in P, then we say that P is *σ-orthocomplete*. For the remainder of this paper we will assume that P has this property.

Let $\mathscr{B}$ denote the Borel subsets of the real line $\mathscr{R}$. By an *observable* (or *P-valued measure*) on $\mathscr{B}$ we mean a mapping

$$A : \mathscr{B} \to P$$

such that

(3) $A(\phi) = 0$, $A(\mathscr{R}) = 1$,

(4) $E, F \in \mathscr{B}$, $E \cap F = \phi \Rightarrow A(E) \perp A(F)$, and

(5) if $\{E_i | i = 1, 2, \ldots\}$ is a family of disjoint subsets of $\mathscr{B}$, then

$$A\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigvee_{i=1}^{\infty} A(E_i)$$

We denote the set of all observables by $\mathscr{O}$.

By a *state* on an orthomodular poset P we mean a function

$$\alpha : P \to [0, 1]$$

such that

(6) $\alpha(0) = 0$, $\alpha(1) = 1$, and

(7) if $\{x_i | i = 1, 2, \ldots\}$ is a mutually orthogonal family in P, then

$$\alpha\left(\bigvee_{i=1}^{\infty} x_i\right) = \sum_{i=1}^{\infty} \alpha(x_i)$$

It is easily shown that if $\alpha \circ A = \beta \circ A$ for every $A \in \mathscr{O}$, then $\alpha = \beta$. Note that for any observable A and any state α, $\alpha \circ A : \mathscr{B} \to [0, 1]$ is a Borel probability measure. If $x_1, x_2 \in P$ with $x_1 \leqslant x_2$, then $\alpha(x_1) \leqslant \alpha(x_2)$ for every state α. If $\mathscr{S}$ is any set of states for P we say that $\mathscr{S}$ is *full* providing that $\alpha(x_1) \leqslant \alpha(x_2)\ \forall \alpha \in \mathscr{S} \Rightarrow x_1 \leqslant x_2$. Not all orthomodular posets admit a full set of states, for M. K. Bennett (1968) has shown that G_{32}, the Greechie 32-element lattice, fails to admit a full set of states. We define a *quantum logic* (or simply a *logic*) to be any σ-orthocomplete orthomodular poset L that admits a full set of states. Henceforth the symbol L will represent a logic and $\mathscr{S}$ a full set of states on L. If $A, B \in \mathscr{O}$ and $\alpha \circ A = \alpha \circ B\ \forall \alpha \in \mathscr{S}$, then $A = B$. By the *resolvent* of an observable A we mean the set $r(A)$ defined by

$$r(A) = \cup\{I | I \text{ is an open inverval in } \mathscr{R} \text{ and } (\forall \alpha \in \mathscr{S})(\alpha \circ A(I) = 0)\}$$

By a *spectrum* of A we mean the set

$$s(A) = \mathscr{R} \setminus r(A)$$

The *point spectrum* of A is the set

$$p(A) = \{x \in \mathscr{R} | \exists \alpha \in \mathscr{S} \text{ such that } \alpha \circ A(\{x\}) \neq 0\}$$

It is easy to see that $p(A) \subset s(A)$. The *continuous spectrum* of A is the set

$$c(A) = s(A) \setminus p(A).$$

2.2. *Theorem*

Let $A \in \mathcal{O}$. Then

(i) $s(A)$ is closed.
(ii) $x \in s(A) \Leftrightarrow (\forall\, \epsilon > 0)(\exists\, \alpha \in \mathscr{S})(\alpha \circ A(x - \epsilon, x + \epsilon) \neq 0)$.
(iii) $(\forall\, \alpha \in \mathscr{S})(\alpha \circ A(s(A)) = 1$ and $\alpha \circ A(r(A)) = 0)$.
(iv) $s(A) = \bigcap\{E | E$ is closed and $(\forall\, \alpha \in \mathscr{S})(\alpha \circ A(E) = 1)\}$.

Proof: (i) is clear

(ii) $x \in s(A) \Leftrightarrow x \notin r(A) \Leftrightarrow \forall$ open interval I containing x $\exists$ an $\alpha \in \mathscr{S}$ such that $\alpha \circ A(I) \neq 0 \Leftrightarrow (\forall\, \epsilon > 0)(\exists\, \alpha \in \mathscr{S})$ such that $\alpha \circ A(x - \epsilon, x + \epsilon) \neq 0$.

(iii) $\alpha \circ A(s(A)) = 1 - \alpha \circ A(r(A))$. By the structure of open sets in $\mathscr{R}$ we can find a countable collection $\{I_K\}$ of open intervals such that

$$r(A) = \bigcup_{K=1}^{\infty} I_K$$

Hence

$$1 \geqslant \alpha \circ A(s(A)) = 1 - \alpha \circ A(r(A)) = 1 - \alpha \circ A\left(\bigcup_{K=1}^{\infty} I_K\right) \geqslant 1 - \sum_{K=1}^{\infty} \alpha \circ A(I_K) = 1$$

(iv) Let E be a closed subset of $\mathscr{R}$ such that $(\forall\, \alpha)(\alpha \circ A(E) = 1)$. Then $\mathscr{R} \setminus E$ is open and $(\forall\, \alpha)(\alpha \circ A(\mathscr{R} \setminus E)$. Clearly $\mathscr{R} \setminus E \subset r(A) = \mathscr{R} \setminus s(A)$ so $s(A) \subset E$.

A subset B of L is said to be a *Boolean subalgebra* (resp. *σ-subalgebra*) of L providing

(8) B is closed under joins (resp. countable joins) in L,
(9) B is closed under $'$, and
(10) B is a Boolean algebra with respect to joins, meets, and $'$ in L.

It is easily seen that if B is a Boolean subalgebra of L, then $x, y \in B \Rightarrow x \mathsf{C}\, y$. It is a straightforward exercise, involving Theorem 2.1, to show that for every $A \in \mathcal{O}$, $A(\mathscr{B})$ is a Boolean σ-subalgebra of L.

3. *Spectral Mapping Theorems and the Observable $A - \lambda I$*

If the logic L is taken to be the projection lattice of a complex Hilbert space $\mathscr{H}$, then via the spectral theorem, the set of observables can be identified with the self adjoint operators on $\mathscr{H}$. One can then classify the spectra of A by considering the character of $(A - \lambda I)^{-1}$. In this section we will show that this classification procedure can be done completely without the aid of Hilbert space formalism. In establishing this result we will have occasion to prove some spectral mapping theorems.

Let f be a real valued function whose domain, $\operatorname{dom} f$, is a subset of the reals. We say that f is a *Borel function* providing $\operatorname{dom} f \in \mathscr{B}$ and for each $G \in \mathscr{B}$, $f^{-1}(G) \in \mathscr{B}$. Clearly any Borel function f can be extended to a Borel function $\hat{f}$ where $\operatorname{dom} \hat{f} = \mathscr{R}$. (Just define $\hat{f}(x) = 0$ for $x \in \mathscr{R} \setminus \operatorname{dom} f$.) If f is a Borel function with $\operatorname{dom} f = \mathscr{R}$ and if $A \in \mathcal{O}$, we define $f(A)$ to be the observable $A \circ f^{-1}$. If $\operatorname{dom} f \neq \mathscr{R}$ we say that $f(A)$ *exists* or *is defined* providing it is the case that for every pair of Borel extensions f_1, f_2 of f with $\operatorname{dom}(f_1) = \operatorname{dom}(f_2) = \mathscr{R}$ we have $f_1(A) = f_2(A)$. If $f(A)$ exists we define $f(A)$ to be $\hat{f}(A)$ for any extension $\hat{f}$ of f with $\operatorname{dom}(\hat{f}) = \mathscr{R}$.

3.1. *Theorem*

Let $A \in \mathcal{O}$, f a Borel function. Then $f(A)$ exists $\Leftrightarrow \alpha \circ A(\operatorname{dom} f) = 1$ for all $\alpha \in \mathscr{S}$.

Proof: If $f(A)$ exists, let $y_2 \in f(\operatorname{dom} f)$ and let $y_1 \neq y_2$. Define

$$f_i(x) = \begin{cases} f(x) \text{ for } x \in \operatorname{dom} f \\ y_i \text{ for } x \in \mathscr{R} \setminus \operatorname{dom} f \end{cases} \qquad (i = 1 \text{ or } 2)$$

Each f_i is a Borel extension of f. Thus for each $\alpha \in \mathscr{S}$,

$$\alpha \circ A \circ f_1^{-1}(\{y_2\}) = \alpha \circ A \circ f_2^{-1}(\{y_2\})$$

so

$$\alpha \circ A(f_1^{-1}(\{y_2\})) = \alpha \circ A(f_2^{-1}(\{y_2\}))$$

But

$$f_1^{-1}(\{y_2\}) = f^{-1}(\{y_2\}) \quad \text{and} \quad f_2^{-1}(\{y_2\}) = (\mathscr{R} \setminus \operatorname{dom} f) \cup f^{-1}(\{y_2\})$$

Thus

$$\alpha \circ A(f^{-1}(\{y_2\})) = \alpha \circ A(\mathscr{R} \setminus \operatorname{dom} f) + \alpha \circ A(f^{-1}(\{y_2\}))$$

whence

$$\alpha \circ A(\mathscr{R}\setminus \operatorname{dom} f) = 0$$

Therefore

$$\alpha \circ A(\operatorname{dom} f) = 1$$

for each $\alpha \in \mathscr{S}$.

Conversely, suppose that $\alpha \circ A(\operatorname{dom} f) = 1$ for every $\alpha \in \mathscr{S}$ so that $\alpha \circ A(\mathscr{R}\setminus \operatorname{dom} f) = 0$ for every $\alpha \in \mathscr{S}$. Let f_1, f_2 be Borel extensions of f. Then

$$f_1^{-1}(E) \cap \operatorname{dom} f = f_2^{-1}(E) \cap \operatorname{dom} f$$

for all $E \in \mathscr{B}$. Now $\forall\ \alpha \in \mathscr{S}$, $\forall\ E \in \mathscr{B}$,

$$\begin{aligned}\alpha \circ (f_i(A))(E) &= \alpha \circ A(f_i^{-1}(E))\\ &= \alpha \circ A(f_i^{-1}(E) \cap ((\mathscr{R}\setminus \operatorname{dom} f) \cup \operatorname{dom} f))\\ &= \alpha \circ A(f_i^{-1}(E) \cap (\mathscr{R}\setminus \operatorname{dom} f)) + \alpha \circ A(f_i^{-1}(E) \cap \operatorname{dom} f)\\ &= \alpha \circ A(f_i^{-1}(E) \cap \operatorname{dom} f)\end{aligned}$$

Thus

$$\alpha \circ f_1(A) = \alpha \circ f_2(A)$$

Since $\mathscr{S}$ is full,

$$f_1(A) = f_2(A)$$

3.2. *Theorem*

Let $A \in \mathscr{O}$ and suppose that for a Borel function f, $f(A)$ is defined.

(i) $\alpha \circ A(\operatorname{dom} f \cap s(A)) = 1\ \forall\ \alpha \in \mathscr{S}$.

(ii) $s(A) \subset \overline{\operatorname{dom} f}$.

(iii) $f(A) = A \circ f^{-1}$.

(iv) If $g(f(A))$ is defined, then so is $(g \circ f)(A)$ and $g(f(A)) = (g \circ f)(A)$.

Proof: (i)

$$\begin{aligned}1 &= \alpha \circ A(\operatorname{dom} f \cup s(A)) = \alpha \circ A(\operatorname{dom} f \setminus s(A))\\ &\quad + \alpha \circ A(\operatorname{dom} f \cap s(A)) + \alpha \circ A(s(A)\setminus \operatorname{dom} f)\\ &= \alpha \circ A(\operatorname{dom} f \cap s(A))\end{aligned}$$

(ii) By Theorem 3.1, $\alpha \circ A(\operatorname{dom} f) = 1$ for every $\alpha \in \mathscr{S}$. Hence $\alpha \circ A(\overline{\operatorname{dom} f}) = 1$ for every $\alpha \in \mathscr{S}$. Now apply 2.2(iv).

(iii) Let $\hat{f}$ be any extension of f. Then

$$\forall\ \alpha \in \mathscr{S}, \qquad \forall\ E \in \mathscr{B},$$

$$\begin{aligned}\alpha \circ f(A)(E) &= \alpha \circ \hat{f}(A)(E) = \alpha \circ A(\hat{f}^{-1}(E))\\ &= \alpha \circ A(\hat{f}^{-1}(E) \cap \operatorname{dom} f)\\ &\quad + \alpha \circ A(\hat{f}^{-1}(E) \cap (\mathscr{R}\setminus \operatorname{dom} f))\\ &= \alpha \circ A(\hat{f}^{-1}(E) \cap \operatorname{dom} f) = \alpha \circ A(f^{-1}(E))\end{aligned}$$

(iv) This follows at once from (iii).

3.3. *Lemma*

Let $A \in \mathcal{O}$, and let f be a Borel function such that $f(A)$ is defined. Then

$$s(f(A)) \subset \overline{f(s(A))}$$

Proof:

$$\alpha \circ f(A)(\overline{f(s(A))}) = \alpha \circ A(f^{-1}(\overline{f(s(A))})) \geqslant \alpha \circ A(\operatorname{dom} f \cap s(A)) = 1.$$

Hence $\forall\ \alpha \in \mathscr{S}$, $\alpha \circ f(A)(\overline{f(s(A))}) = 1$ and so by 2.2(iv), $s(f(A)) \subset \overline{f(s(A))}$.

3.4. *Theorem*

If f is continuous on $s(A)$, or if f has a continuous extension to $s(A)$, and if $f(A)$ exists, then

$$s(f(A)) = \overline{f(s(A))}.$$

Proof: By 3.3 it suffices to prove that $\overline{f(s(A))} \subset s(f(A))$. If $\hat{f}$ is a continuous extension of f to $s(A)$, then $\hat{f}(A) = f(A)$ and $\overline{f(s(A))} \subset \overline{\hat{f}(s(A))}$. Thus it would suffice to show in this case that $\overline{\hat{f}(s(A))} \subset s(\hat{f}(A))$. In other words, we can suppose that f is defined and continuous on all of $s(A)$.

Let $y \in \overline{f(s(A))}$. Then there exists a sequence $\{x_i\} \subset s(A)$ such that $f(x_i) \to y$. By 2.2(ii), we have that $\forall\ \delta > 0$ there exists $\alpha \in \mathscr{S}$ such that $\alpha \circ A(x_i - \delta,\, x_i + \delta) \neq 0$. Therefore, by continuity, $\forall\ \epsilon > 0$, there exists $\alpha \in \mathscr{S}$ such that

$$\begin{aligned}\alpha \circ f(A)(f(x_i) - \epsilon, f(x_i) + \epsilon) &= \alpha \circ A(f^{-1}(f(x_i) - \epsilon, f(x_i) + \epsilon)) \\ &\geqslant \alpha \circ A((x_i - \delta_f(\epsilon, x_i),\, x_i + \delta_f(\epsilon, x_i)) \cap s(A) \neq 0\end{aligned}$$

Thus by 2.2(ii), $f(x_i) \in s(f(A))\ \forall\, i$. Since $s(f(A))$ is closed, $y \in s(f(A))$.

An observable A is said to be *bounded* providing that $s(A)$ is compact. For bounded observables we obtain the following.

3.5. *Corollary* (*Spectral Mapping Theorem*)

If A is a bounded observable, f a Borel function defined and continuous on $s(A)$, then if $f(A)$ exists, $s(f(A)) = f(s(A))$.

$A \in \mathcal{O}$ is said to be *invertible* providing $f(A)$ exists for the function $f(x) = 1/x$. In this case we write $A^{-1} = f(A)$. According to Theorem 3.1, A is invertible if and only if $\alpha \circ A(\{0\}) = 0\ \forall\ \alpha \in \mathscr{S}$. In particular, if $0 \notin s(A)$, then A^{-1} exists.

3.6. *Theorem*

Let $A \in \mathcal{O}$ be invertible. Then

(i) $(A^{-1})^{-1}$ exists and $(A^{-1})^{-1} = A$.
(ii) $0 \notin s(A) \Rightarrow A^{-1}$ is bounded.
(iii) A bounded $\Rightarrow 0 \notin s(A^{-1})$.

Proof: (i) Let

$$f(x) = \begin{cases} 1/x \text{ for } x \neq 0 \\ 0 \text{ for } x = 0 \end{cases}$$

Then

$$A^{-1} = f(A). \text{ Now } \alpha \circ A^{-1}(\{0\}) = \alpha \circ f(A)(\{0\}) = \alpha \circ A(f^{-1}(\{0\})) = \alpha \circ A(\{0\}) = 0 \ \forall \ \alpha \in \mathscr{S}$$

whence A^{-1} is invertible. Thus

$$(A^{-1})^{-1} = f(f(A)) = (f \circ f)(A) = A$$

(ii) $0 \notin s(A) \Rightarrow$ there exists an open interval $I = (-\gamma, \gamma)$ such that $0 \in I \subset r(A)$. Thus $\mathscr{R} \setminus I \supset \mathscr{R} \setminus r(A) = s(A)$. Since $0 \notin s(A)$, $f(x) = 1/x$ is continuous on $s(A)$ and so by Theorem 3.4,

$$s(A^{-1}) = \overline{f(s(A))} \subset \overline{f(\mathscr{R} \setminus I)} = \left[-\frac{1}{\gamma}, \frac{1}{\gamma}\right]$$

Thus $s(A^{-1})$ is compact.

The proof of (iii) is similar to (ii).

If we define $f_\lambda : \mathscr{R} \to \mathscr{R}$ by $f_\lambda(x) = x - \lambda$, then it is natural to write $A - \lambda = f_\lambda(A)$. We now show that spectra can be classified using $A - \lambda$ in exactly the same manner as is usually done in operator theory.

3.7. *Theorem*

Let $A \in \mathcal{O}$.

(i) $\lambda \in r(A) \Leftrightarrow (A - \lambda)^{-1}$ exists and is bounded.
(ii) $\lambda \in p(A) \Leftrightarrow (A - \lambda)^{-1}$ does not exist.
(iii) $\lambda \in c(A) \Leftrightarrow (A - \lambda)^{-1}$ exists and is not bounded.

Proof: We first observe that $\forall \ \lambda \ f_\lambda$ is continuous on $s(A)$. It follows that

$$s(A - \lambda) = s(A) - \{\lambda\}$$

(i) $(A - \lambda)^{-1}$ exists and is bounded

$$\Leftrightarrow 0 \notin s(A - \lambda) \Leftrightarrow 0 \notin s(A) - \{\lambda\} \Leftrightarrow \lambda \notin s(A) \Leftrightarrow \lambda \in r(A)$$

(ii) $(A - \lambda)^{-1}$ fails to exist $\Leftrightarrow \exists \ \alpha \in \mathscr{S}$ such that $\alpha \circ A(f_\lambda^{-1}(\{0\}) \neq 0 \Leftrightarrow \exists \ \alpha \in \mathscr{S}$ such that $\alpha \circ A(\{\lambda\}) \neq 0 \Leftrightarrow \lambda \in p(A)$.

(iii) $\lambda \in c(A) \Leftrightarrow$ (i) and (ii) fail.

4. *The Spectral Theorem*

As mentioned before, if L is the projection lattice of a Hilbert space $\mathscr{H}$, then the observables can be identified with the set of self-adjoint operators on $\mathscr{H}$. Then, via the spectral theorem for self-adjoint operators, there is a one-to-one correspondence between observables and spectral resolutions of the identity. In this section we will show that the same result is possible for any logic L. The author would like to thank D. J. Foulis who showed him the details of this section.

By a *real* (respectively *rational*) *spectral resolution* in L we mean a function $e:\mathscr{R} \to L$ (resp. $e:\mathscr{Q} \to L$ where $\mathscr{Q}$ is the set of rational numbers) such that the following conditions are satisfied

(11) $\lambda \leqslant \mu \Rightarrow e_\lambda \leqslant e_\mu$,
(12) $\wedge_\lambda e_\lambda = 0$,
(13) $\vee_\lambda e_\lambda = 1$,
(14) $\bigwedge_{\mu<\lambda} e_\lambda = e_\mu$ for all $\mu \in \mathscr{R}$ (resp. $\mathscr{Q}$).

It is quite clear that if $A \in \mathscr{O}$, then the function $e^A:\mathscr{R} \to L$ defined by $e_\lambda^A = A((-\infty, \lambda])$ is a spectral resolution in L. On the other hand, it e_λ is a spectral resolution in L, does there exist an A such that $e_\lambda = e_\lambda^A$? We will show that providing the range of e_λ is contained in a Boolean σ-subalgebra of L, that the answer is yes. This last requirement is a reasonable one since for any $A \in \mathscr{O}$, range A is a Boolean σ-subalgebra of L. Thus our problem is reduced to proving the following lemma.

4.1. *Lemma*

Let B be a Boolean σ-algebra. Then if e is any spectral resolution in B, there exists a B-valued Borel measure A such that $e_\lambda = A((-\infty, \lambda])$.

The proof of this lemma involves a construction that hinges on the following obvious but crucial lemma.

4.2. *Lemma*

Let B be a Boolean σ-algebra. Then there is a one-to-one correspondence between real and rational spectral resolutions on B as follows:

(i) if $e:\mathscr{R} \to B$ is a real spectral resolution, then the rational spectral resolution f associated with it is given by $f = e|_{\mathscr{Q}}$, and
(ii) if $f:\mathscr{Q} \to B$ is a rational spectral resolution, then the real spectral resolution e associated with it is given by

$$e_\lambda = \wedge\{f_\mu | \mu \in \mathscr{Q}, \lambda \leqslant \mu\}$$

For the remainder of this section we shall suppose that B is a Boolean σ-algebra. By Loomis Theorem (Halmos, 1950, p. 171, 15c) there exists a measurable space $(X, \mathscr{M})$ and a σ-ideal $\mathscr{N} \subset \mathscr{M}$ such that $B \simeq \mathscr{M}/\mathscr{N}$. We propose to identify B with $\mathscr{M}/\mathscr{N}$ and to let $\eta: \mathscr{M} \to B$ be the natural epimorphism. Now suppose $e: \mathscr{R} \to B$ is a real spectral resolution and let $f: \mathscr{Q} \to B$ be the restriction of e to $\mathscr{Q}$. For each rational number $\lambda \in \mathscr{Q}$ choose a set $\tilde{F}_\lambda \in \mathscr{M}$ such that $\eta(\tilde{F}_\lambda) = f_\lambda$. Then for each $\lambda \in \mathscr{Q}$ define

$$\bar{F}_\lambda = \cap\{\tilde{F}_\rho | \lambda < \rho, \rho \in \mathscr{Q}\}$$

It is easily seen that if $\lambda, \mu \in \mathscr{Q}$ with $\lambda \leqslant \mu$, then $\bar{F}_\lambda \subset \bar{F}_\mu$. Also a simple computation shows that for every $\lambda \in \mathscr{Q}$, $\eta(\bar{F}_\lambda) = f_\lambda$. Now define

$$\hat{F}_\lambda = \overline{F}_\lambda \diagdown \left(\bigcap_{\sigma \in \mathscr{Q}} \bar{F}_\sigma \right)$$

It then follows that $\lambda, \mu \in \mathscr{Q}$ with $\lambda \leqslant \mu$ that $\hat{F}_\lambda \subset \hat{F}_\mu$. Also, $\bigcap_{\lambda \in \mathscr{Q}} \hat{F}_\lambda = \emptyset$ and $\eta(\hat{F}_\lambda) = f_\lambda$. Finally, define

$$F_\lambda = \begin{cases} \hat{F}_\lambda \text{ if } \lambda < 0, \lambda \in \mathscr{Q} \\ \hat{F}_\lambda \cup \left(X \diagdown \bigcup_{\sigma \in \mathscr{Q}} \hat{F}_\sigma \right) & \text{if } 0 \leqslant \lambda, \lambda \in \mathscr{Q} \end{cases}$$

4.3. *Lemma*

$\{F_\lambda | \lambda \in \mathscr{Q}\}$ is a rational spectral resolution in $\mathscr{M}$ and $\eta(F_\lambda) = f_\lambda$ for all $\lambda \in \mathscr{Q}$.

Proof: Conditions (11), (12), and (14) are clearly satisfied.

$$\begin{aligned} \vee_\lambda F_\lambda = \cup_\lambda F_\lambda &= \left(\bigcup_{\lambda < 0} \hat{F}_\lambda \right) \cup \left(\bigcup_{0 \leqslant \lambda} \left(\hat{F}_\lambda \cup \left(X \diagdown \bigcup_{\sigma \in \mathscr{Q}} \hat{F}_\sigma \right) \right) \right) \\ &= \left(\bigcup_{\lambda < 0} \hat{F}_\lambda \right) \cup \left(\bigcup_{0 \leqslant \lambda} \hat{F}_\lambda \right) \cup \left(X \diagdown \bigcup_{\sigma \in \mathscr{Q}} \hat{F}_\sigma \right) \\ &= \left(\bigcup_{\lambda \in \mathscr{Q}} \hat{F}_\lambda \right) \cup \left(X \diagdown \bigcup_{\sigma \in \mathscr{Q}} F_\sigma \right) = X \end{aligned}$$

and so (13) holds also.

If $\lambda < 0$, then $\eta(F_\lambda) = f_\lambda$ is clear. If $0 \leqslant \lambda$, then

$$\begin{aligned} \eta(F_\lambda) &= \eta\left(\hat{F}_\lambda \cup \left(X \diagdown \bigcup_{\sigma \in \mathscr{Q}} \hat{F}_\sigma \right) \right) = \eta(\hat{F}_\lambda) \vee \eta\left(X \diagdown \bigcup_{\sigma \in \mathscr{Q}} \hat{F}_\sigma \right) \\ &= f_\lambda \vee \left(\eta\left(\bigcup_{\sigma \in \mathscr{Q}} \hat{F}_\sigma \right) \right)' = f_\lambda \vee \left(\bigvee_{\sigma \in \mathscr{Q}} \eta\left(\hat{F}_\sigma \right) \right)' = f_\lambda \vee \left(\bigvee_{\sigma \in \mathscr{Q}} f_\sigma \right)' \\ &= f_\lambda \vee 1' = f_\lambda \vee 0 = f_\lambda \end{aligned}$$

In any case, $\eta(F_\lambda) = f_\lambda$.

We next apply Lemma 4.2 to the rational spectral resolution F_λ and obtain a real spectral resolution $E:\mathscr{R} \to \mathscr{M}$. It is easily checked that $\eta(E_\lambda) = e_\lambda$.

4.4. *Lemma*

Let $(X,\mathscr{M})$ be any measurable space and let $\{E_\lambda | \lambda \in \mathscr{R}\}$ be any spectral resolution in $\mathscr{M}$. Then there exists a unique measurable function $f:X \to \mathscr{R}$ (measurable in the sense that $E \in \mathscr{R} \Rightarrow f^{-1}(E) \in \mathscr{M}$) such that

$$E_\lambda = f^{-1}((-\infty, \lambda])$$

Proof: Define $f(x) = \inf\{\lambda \in \mathscr{R} | x \in E_\lambda\}$. It is then a straightforward argument to show that $f^{-1}((-\infty,\lambda]) = E_\lambda$. Since f^{-1} preserves complements and countable unions it follows that f is measurable.

Lemma 4.4 essentially concludes our construction. It follows from this that if $e:\mathscr{R} \to B$ is a real spectral resolution that there exists a unique B-valued measure $A:\mathscr{B} \to B$ such that $A((-\infty,\lambda]) = e_\lambda \ \forall \ \lambda \in \mathscr{R}$. To see this, lift e to a spectral resolution $\{E_\lambda | \lambda \in \mathscr{R}\}$ in $\mathscr{M}$. Construct f as in 4.4 and let $A = \eta \circ f^{-1}$.

All of the following can be collected in the following theorem.

4.5. *Theorem* (*The Spectral Theorem*)

Let $e:\mathscr{R} \to L$ be a real spectral resolution, B a σ-subalgebra of L containing range e. Then there exists a unique observable A on L such that

(i) range $A \subset B$,
(ii) $A((-\infty,\lambda]) = e_\lambda$.

Conversely, if A is any observable on L, then range A is a Boolean σ-subalgebra of L and $e_\lambda = A((-\infty,\lambda])$ defines a real spectral resolution in L.

5. *Expectation*

For each observable A and each state $\alpha, \alpha \circ A$ is a Borel probability measure. Thus, one can define a real number called the *expectation of A in the state* α (when it exists), written $\exp_\alpha(A)$, by the integral

$$\exp_\alpha(A) = \int_{\mathscr{R}} id(\alpha \circ A)$$

where i is the identity function. If e_λ is the spectral resolution associated with A, then this can be written as the Stieltjes integral

$$\exp_\alpha(A) = \int_{-\infty}^{\infty} \lambda d\alpha(e_\lambda).$$

If A is bounded then it is clear that $\exp_\alpha(A)$ exists for all states α. If A is unbounded, $\exp_\alpha(A)$ may not exist, e.g., the position observable in ordinary quantum mechanics. Nevertheless, we can establish the following.

5.1. *Theorem*

Let A be any observable, α any state, and f any Borel function such that

(i) $f(A)$ exists, and

(ii) $\int_{\mathscr{R}} f d\alpha \circ A$ exists.

Then $\exp_\alpha(f(A))$ exists and moreover

$$\exp_\alpha(f(A)) = \int_{\operatorname{dom} f \cap s(A)} f d(\alpha \circ A)$$

Proof: Let

$$g(x) = \begin{cases} f(x) & x \in \operatorname{dom} f \\ 0 & x \notin \operatorname{dom} f \end{cases}$$

Then

$$\exp_\alpha(f(A)) = \exp_\alpha(g(A)) = \int_{\mathscr{R}} id(\alpha \circ A \circ g^{-1})$$

$$= \int_{s(g(A))} id(\alpha \circ A \circ g^{-1}) = \int_{\overline{g(s(A))}} id(\alpha \circ A \circ g^{-1})$$

since by 3.3,

$$s(g(A)) \subset g(s(A)) \qquad \text{and} \qquad (\alpha \circ g(A))[\overline{g(s(A))} \setminus s(g(A))] = 0$$

Thus by Halmos (1950, p. 163, Theorem C),

$$\exp_\alpha(f(A)) = \int_{g^{-1}\overline{(g(s(A)))}} g d(\alpha \circ A)$$

But since

$$s(A) \subset g^{-1}(g(s(A))) \subset g^{-1}\overline{(g(s(A)))}$$

and

$$(\alpha \circ A)(g^{-1}\overline{(g(s(A)))} \setminus s(A)) = 0$$

we have

$$\exp_\alpha(f(A)) = \int_{s(A)} g d(\alpha \circ A) = \int_{\operatorname{dom} f \cap s(A)} f d(\alpha \circ A)$$

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